

# Majority-vote model on $(3, 4, 6, 4)$ and $(3^4, 6)$ Archimedean lattices

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On Archimedean lattices, the Ising model exhibits spontaneous ordering. Two examples of these lattices of the majority-vote model with noise are considered and studied through extensive Monte Carlo simulations. The order/disorder phase transition is observed in this system. The calculated values of the critical noise parameter are  $q_c = 0.091(2)$  and  $q_c = 0.134(3)$  for  $(3, 4, 6, 4)$  and  $(3^4, 6)$  Archimedean lattices, respectively. The critical exponents  $\beta/\nu$ ,  $\gamma/\nu$  and  $1/\nu$  for this model are  $0.103(6)$ ,  $1.596(54)$ ,  $0.872(85)$  for  $(3, 4, 6, 4)$  and  $0.114(3)$ ,  $1.632(35)$ ,  $0.98(10)$  for  $(3^4, 6)$  Archimedean lattices. These results differs from the usual Ising model results and the majority-vote model on so-far studied regular lattices or complex networks. The effective dimensionalities of the system [ $D_{\text{eff}}(3, 4, 6, 4) = 1.802(55)$  and  $D_{\text{eff}}(3^4, 6) = 1.860(34)$ ] for these networks are reasonably close to the embedding dimension two.

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## I. INTRODUCTION

The majority-vote model (MVM) [1] defined on regular lattices shows second-order phase transition with critical exponents  $\beta$ ,  $\gamma$ ,  $\nu$  — which characterize the system in the vicinity of the phase transition — identical [1, 2, 3] with those of equilibrium Ising model [4, 5].

On the other hand MVM on the complex networks exhibit different behavior [6, 7, 8, 9]. Campos *et al.* investigated MVM on small-world network [6]. This network was constructed using the square lattice (SL) by the rewiring procedure. Campos *et al.* found that the critical exponents  $\gamma/\nu$  and  $\beta/\nu$  are different from these of the Ising model [5] and depend on the rewiring probability. Pereira *et al.* [7] studied MVM on Erdős–Rényi’s (ER) classical random graphs [10], and Lima *et al.* [8] also studied this model on random Voronoy–Delaunay [13] lattice with periodic boundary conditions. Very recently Lima [9] studied the MVM on directed Albert–Barabási (AB) network [11] and contrary to the Ising model [4] on these networks [12], the order/disorder phase transition is observed in this system. The calculated  $\beta/\nu$  and  $\gamma/\nu$  exponents are different from those for the Ising model [5] and depend on the mean value of connectivity  $z$  of AB network. The latter was observed also for ER random graph [7].

The results obtained by these authors [6, 7, 8, 9] show that the MVM on various complex topologies belongs to different universality classes. Moreover, contrary to MVM on regular lattices [1, 2], the obtained critical exponents are different from those of the universality class to which the equilibrium Ising model belongs [5].

In this paper we study the MVM on two Archimedean lattices (AL), namely on  $(3, 4, 6, 4)$  and  $(3^4, 6)$ . The topologies of  $(3, 4, 6, 4)$  and  $(3^4, 6)$  AL are presented in Fig. 1. The AL are vertex transitive graphs that can be embedded in a plane such that every face is a regular polygon. Kepler showed that there are exactly eleven such graphs. The AL are labeled according to the sizes of faces incident to a given vertex. The face sizes are sorted, starting from the face for which the list is the smallest in lexicographical order. In this way, the square lattice gets the name  $(4, 4, 4, 4)$ , abbreviated to  $(4^4)$ , honeycomb is called  $(6^3)$  and Kagomé is  $(3, 6, 3, 6)$ . Critical properties of these lattices were investigated in terms of site percolation [14] in Ref. [15]. Topologies of all eleven AL are given there as well. Very recently, the Ising model on those AL was investigated in Ref. [16].

Our main goal is to check the hypothesis of Grinstein *et al.* [17] — i.e., that non-equilibrium stochastic spin systems with up-down symmetry fall in the universality class of the equilibrium Ising model — for systems in-between ordinary, regular lattices (like square one [1]) and complex spin systems (like decorated with spin variables classical random graphs [7] or scale-free networks [9]).

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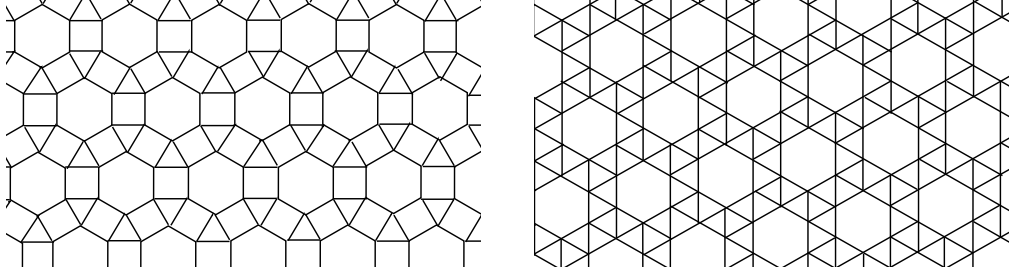


FIG. 1:  $(3, 4, 6, 4)$  (left) and  $(3^4, 6)$  (right) AL.

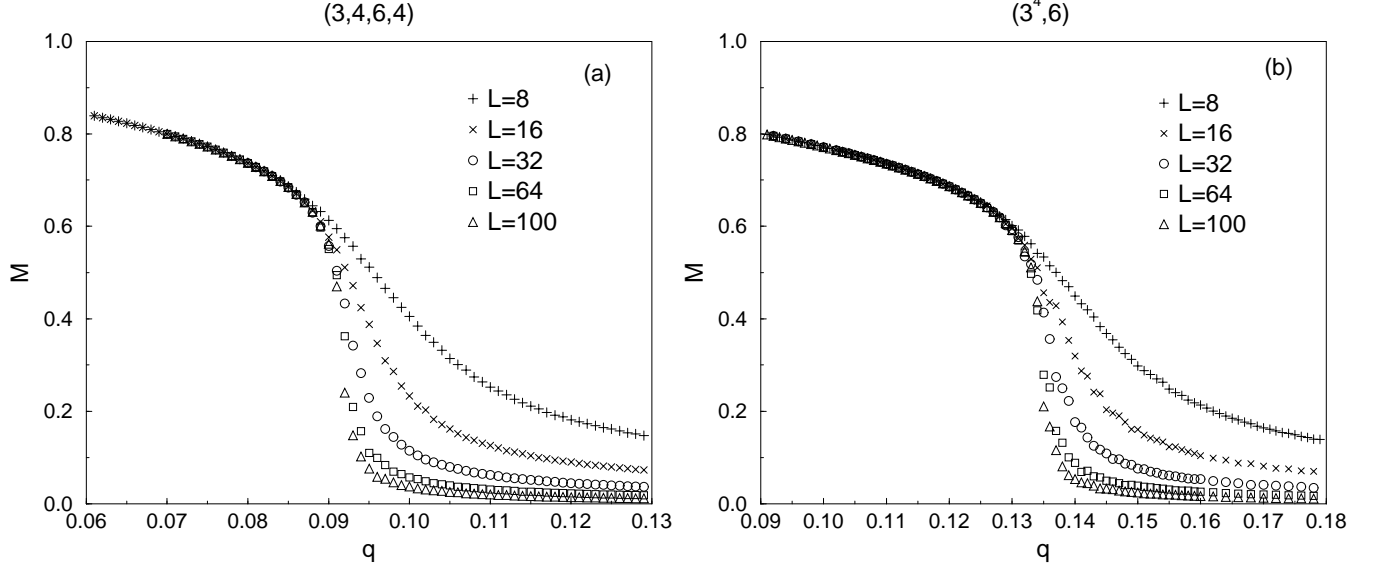


FIG. 2: The magnetization  $M$  as a function of the noise parameter  $q$ , for  $L = 8, 16, 32, 64$  and  $100$  sites for (a)  $(3, 4, 6, 4)$  and (b)  $(3^4, 6)$  AL.

With extensive Monte Carlo simulation we show that MVM on  $(3, 4, 6, 4)$  and  $(3^4, 6)$  *regular* AL exhibits second-order phase transition with effective dimensionality  $D_{\text{eff}} \approx 1.8$  and has critical exponents that *do not* fall into universality class of the equilibrium Ising model (although for  $(3^4, 6)$  lattice they are quite close, yet lower).

## II. MODEL AND SIMULATION

We consider the MVM [1] defined by a set of “voters” or spin variables  $\sigma$  taking the values  $+1$  or  $-1$ , situated on every node of the  $(3, 4, 6, 4)$  and  $(3^4, 6)$  AL with  $N = 6L^2$  sites. The evolution is governed by single spin-flip like dynamics with a probability  $w_i$  of  $i$ -th spin flip given by

$$w_i = \frac{1}{2} \left[ 1 - (1 - 2q) \sigma_i \cdot \text{sign} \left( \sum_{j=1}^z \sigma_j \right) \right], \quad (1)$$

where the sum runs over the number  $z$  ( $z(3, 4, 6, 4) = 4$  or  $z(3^4, 6) = 5$ ) of nearest neighbors of  $i$ -th spin. The control parameter  $0 \leq q \leq 1$  plays the role of the temperature in equilibrium systems and measures the probability of parallel aligning to the majority of neighbors. It means, that given spin  $i$  adopts the majority sign of its neighbors with probability  $q$  and the minority sign with probability  $(1 - q)$  [1, 7, 8, 9].

To study the critical behavior of the model we define the variable  $m \equiv \sum_{i=1}^N \sigma_i / N$ . In particular, we are interested

in the magnetization  $M$ , susceptibility  $\chi$  and the reduced fourth-order cumulant  $U$

$$M(q) \equiv \langle |m| \rangle, \quad (2a)$$

$$\chi(q) \equiv N (\langle m^2 \rangle - \langle m \rangle^2), \quad (2b)$$

$$U(q) \equiv 1 - \langle m^4 \rangle / (3 \langle m^2 \rangle^2), \quad (2c)$$

where  $\langle \dots \rangle$  stands for a thermodynamics average. The results are averaged over the  $N_{\text{run}}$  independent simulations.

These quantities are functions of the noise parameter  $q$  and obey the finite-size scaling relations

$$M = L^{-\beta/\nu} f_m(x), \quad (3a)$$

$$\chi = L^{\gamma/\nu} f_\chi(x), \quad (3b)$$

$$\frac{dU}{dq} = L^{1/\nu} f_U(x), \quad (3c)$$

where  $\nu$ ,  $\beta$ , and  $\gamma$  are the usual critical exponents,  $f_i(x)$  are the finite size scaling functions with

$$x = (q - q_c) L^{1/\nu} \quad (3d)$$

being the scaling variable. Therefore, from the size dependence of  $M$  and  $\chi$  we obtained the exponents  $\beta/\nu$  and  $\gamma/\nu$ , respectively. The maximum value of susceptibility also scales as  $L^{\gamma/\nu}$ . Moreover, the value of  $q^*$  for which  $\chi$  has a maximum is expected to scale with the system size as

$$q^* = q_c + b L^{-1/\nu} \text{ with } b \approx 1. \quad (4)$$

Therefore, the relations (3c) and (4) may be used to get the exponent  $1/\nu$ . We evaluate also the effective dimensionality,  $D_{\text{eff}}$ , from the hyperscaling hypothesis

$$2\beta/\nu + \gamma/\nu = D_{\text{eff}}. \quad (5)$$

We performed Monte Carlo simulation on the  $(3, 4, 6, 4)$  and  $(3^4, 6)$  AL with various systems of size  $N = 384, 1536, 6144, 24576$  and  $60000$ . It takes  $2 \times 10^5$  Monte Carlo steps (MCS) to make the system reach the steady state, and then the time averages are estimated over the next  $2 \times 10^5$  MCS. One MCS is accomplished after all the  $N$  spins are investigated whether they flip or not. We carried out  $N_{\text{run}} = 20$  to  $50$  independent simulation runs for each lattice and for given set of parameters  $(q, N)$ .

### III. RESULTS AND DISCUSSION

In Fig. 2 we show the dependence of the magnetization  $M$  on the noise parameter  $q$ , obtained from simulations on  $(3, 4, 6, 4)$  and  $(3^4, 6)$  AL with  $N$  ranging from  $384$  to  $60000$  sites. The shape of  $M(q)$  curve, for a given value of  $N$ , suggests the presents of the second-order phase transition in the system. The phase transition occurs at the value of the critical noise parameter  $q_c$ .

In Fig. 3 the corresponding behavior of the susceptibility  $\chi$  is presented. In Fig. 4 we plot the Binder's fourth-order cumulant  $U$  for different values of the system size  $N$ . The critical noise parameter  $q_c$  is estimated as the point where the curves for different system sizes  $N$  intercept each other [18]. From Fig. 4 we obtain  $q_c = 0.091(2)$  and  $q_c = 0.134(3)$  for  $(3, 4, 6, 4)$  and  $(3^4, 6)$  AL, respectively.

In Fig. 5 we plot the dependence of the magnetization  $M^* = M(q_c)$  vs. the linear system size  $L$ . The slopes of curves correspond to the exponent ratio  $\beta/\nu$  according to Eq. (3a). The obtained exponents are  $\beta/\nu = 0.103(6)$  and  $0.114(3)$  respectively for  $(3, 4, 6, 4)$  and  $(3^4, 6)$  AL. The exponents ratio  $\gamma/\nu$  are obtained from the slopes of the straight lines with  $\gamma/\nu = 1.596(54)$  for  $(3, 4, 6, 4)$  and  $\gamma/\nu = 1.632(35)$  for  $(3^4, 6)$ , as presented in Fig. 6.

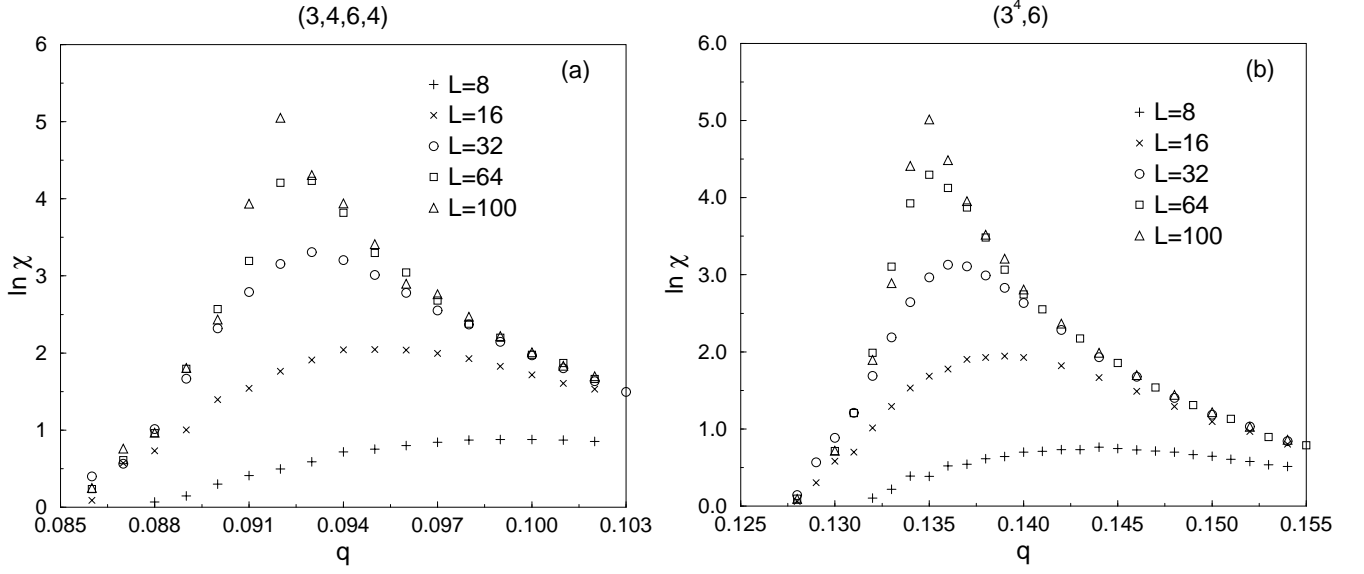


FIG. 3: Susceptibility versus  $q$  for (a)  $(3,4,6,4)$  and (b)  $(3^4,6)$  AL.

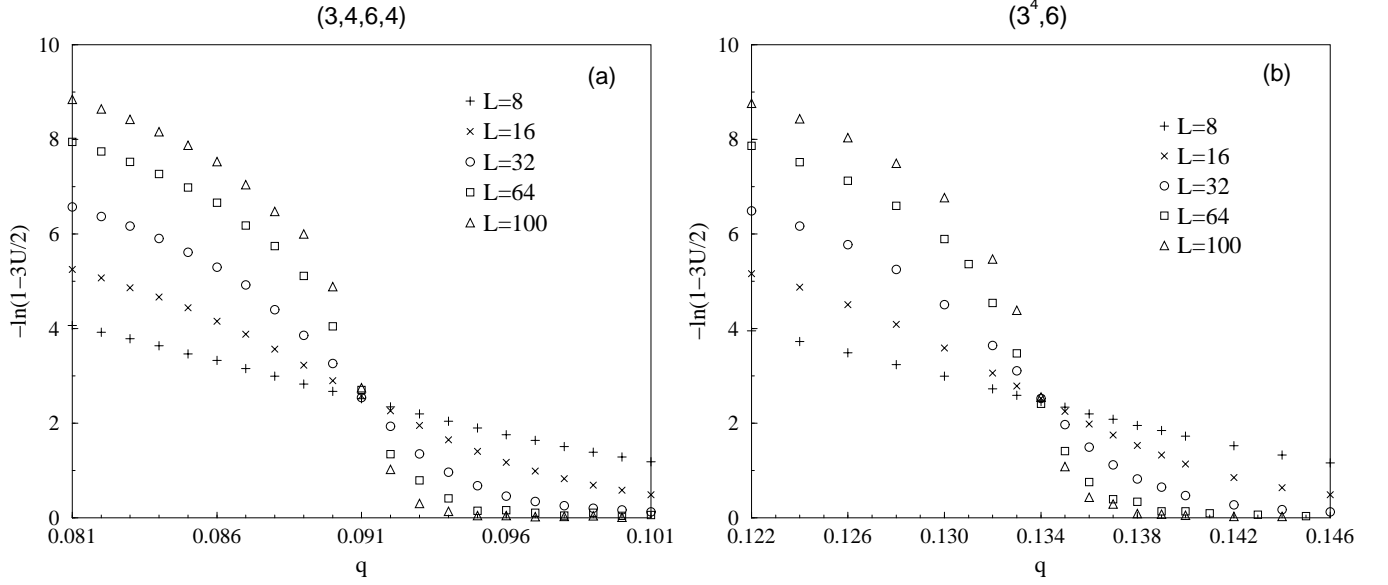


FIG. 4: The reduced Binder's fourth-order cumulant  $U$  as a function of  $q$  for (a)  $(3,4,6,4)$  and (b)  $(3^4,6)$  AL.

To obtain the critical exponent  $1/\nu$ , we used the scaling relation (4). The calculated values of the exponents  $1/\nu$  are 0.872(85) for  $(3,4,6,4)$  (circle) and  $1/\nu = 0.98(10)$  for  $(3^4,6)$  (square) (see Fig. 7). Eq. (5) yields effective dimensionality of systems  $D_{\text{eff}} = 1.802(55)$  for  $(3,4,6,4)$  and  $D_{\text{eff}} = 1.860(34)$  for  $(3^4,6)$ . The MVM on those two AL has the effective dimensionality close to two contrary to ER classical random graphs ( $0.99 \leq D_{\text{eff}} \leq 1.02$ ) [7] or directed AB networks ( $0.998 \leq D_{\text{eff}} \leq 1.018$ ) [9] with roughly the same nodes connectivity ( $\bar{z} = 4$ ) as for  $(3,4,6,4)$  and  $(3^4,6)$  AL.

The results of simulations together with data for ER and AB networks with mean connectivity  $\bar{z} = 4$  are collected in Tab. I.

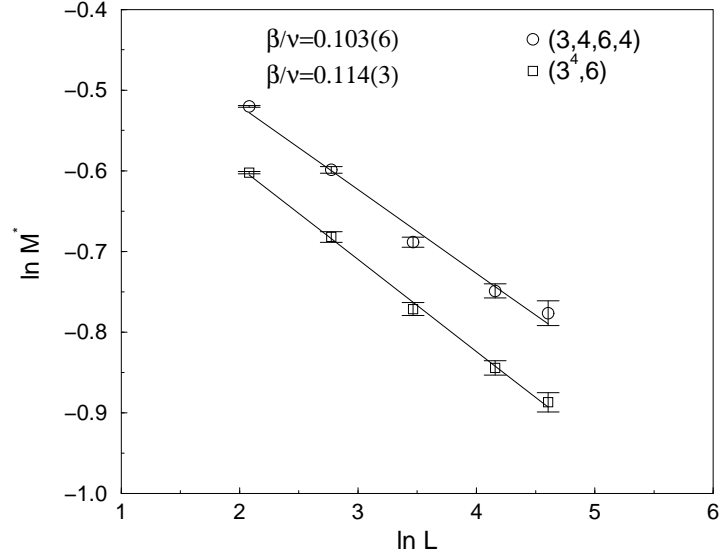


FIG. 5: Plot of  $\ln M^*$  vs.  $\ln L$  for  $(3, 4, 6, 4)$  and  $(3^4, 6)$  AL.

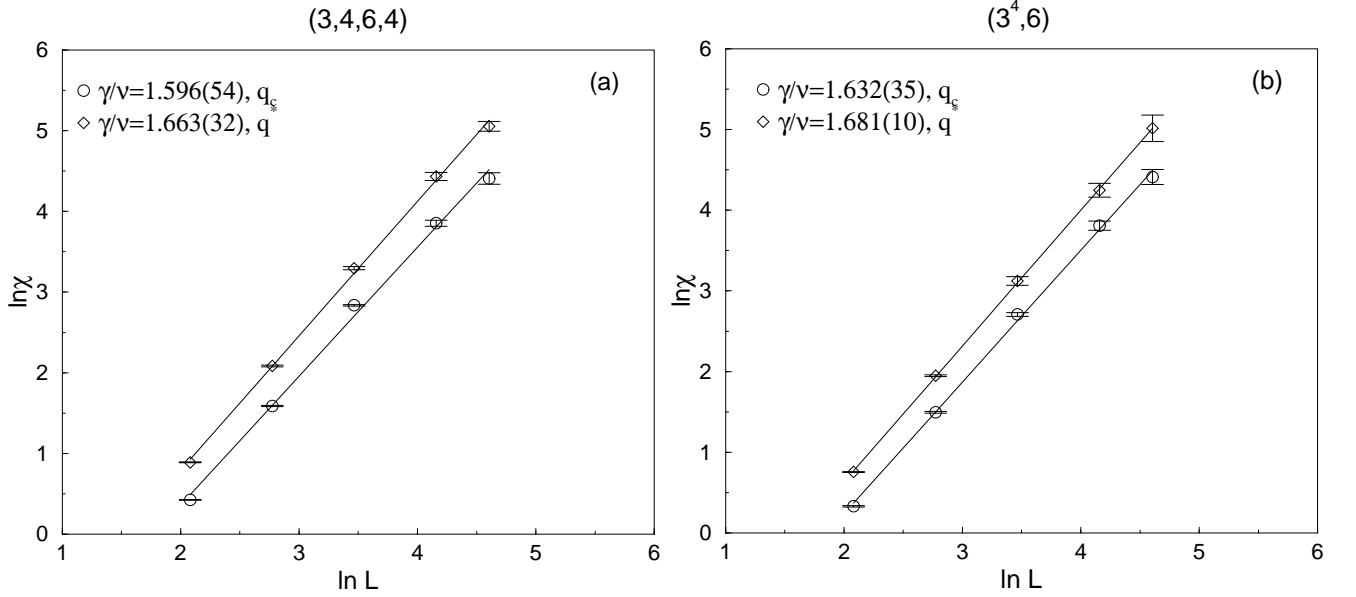


FIG. 6: Critical behavior of the susceptibility  $\chi(N)$  at  $q = q_c$  (circle) and  $q = q^*$  (diamond) for (a)  $(3, 4, 6, 4)$  and (b)  $(3^4, 6)$  AL.

#### IV. CONCLUSION

We presented a very simple non-equilibrium MVM on  $(3, 4, 6, 4)$  and  $(3^4, 6)$  AL. On these lattices, the MVM shows a second-order phase transition. Our Monte Carlo simulations demonstrate that the effective dimensionality  $D_{\text{eff}}$  is close to two, i.e. that hyperscaling may be valid.

Finally, we remark that the critical exponents  $\gamma/\nu$ ,  $\beta/\nu$  and  $1/\nu$  for MVM on *regular*  $(3, 4, 6, 4)$  AL are *different* from the Ising model [5] and *differ* from those for so-far studied regular lattices [1, 2] and for the ER classical random graphs [7] and for the directed AB network [9]. However, in the latter cases [7, 9] the scaling relations (3) must involve the number of sites  $N$  instead of linear system size  $L$  as these networks in natural way do not possess such characteristic which allow for  $N \propto L^d$  ( $d \in \mathbb{Z}$ ) dependence [20]. For  $(3^4, 6)$  AL, the critical exponents are much closer

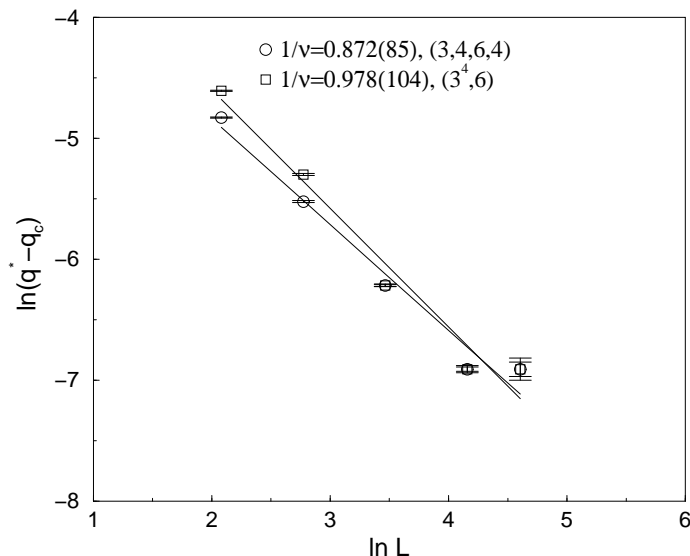


FIG. 7: The exponents  $1/\nu$  obtained from the relation (4) for  $(3, 4, 6, 4)$  (circle) and  $(3^4, 6)$  (square) AL.

TABLE I: Critical parameter, exponents and effective dimension for MVM model on  $(3, 4, 6, 4)$  and  $(3^4, 6)$  AL. For completeness we cite data for scale-free AB and random ER graphs with an average connectivity  $\bar{z} = 4$ .

	SL [1]	AB [9]	ER [7]	$(3,4,6,4)$	$(3^4, 6)$
$q_c$	0.075(10)	0.431(3)	0.181(1)	0.091(2)	0.134(3)
$\beta/\nu$	0.125(5)	0.447(1)	0.242(6)	0.103(6)	0.114(3)
$\gamma/\nu^a$	1.73	0.104(2)	0.54(1)	1.596(54)	1.632(35)
$\gamma/\nu^b$	1.70	0.888(9)	0.515(6)	1.663(32)	1.681(10)
$1/\nu$	1/0.99(5)	—	0.59(7)	0.872(85)	0.98(10)
$D_{\text{eff}}^c$	—	0.998(3)	1.02(2)	1.802(55)	1.860(34)

<sup>a</sup>obtained using  $\chi(N)$  at  $q = q_c$

<sup>b</sup>obtained using  $\chi(N)$  at  $q = q^*$

<sup>c</sup>obtained using ratio  $\gamma/\nu$  given by dependence  $\chi(N)$  at  $q = q_c$

to those known analytically for square lattice Ising model, i.e.  $\beta = 1/8 = 0.125$ ,  $\gamma = 7/4 = 1.75$  and  $\nu = 1$ , but except for  $\nu$  they differ for more than three numerically estimated uncertainties. As those uncertainties were estimated basing only on statistical analysis we cannot excluding that critical exponents may reach SL Ising exponents asymptotically in the thermodynamic limit.

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- [1] M. J. de Oliveira, J. Stat. Phys. **66**, 273 (1992).
  - [2] M. A. Santos, S. Teixeira, J. Stat. Phys. **78**, 963 (1995).
  - [3] L. Crochik, T. Tomé, Phys. Rev. E **72**, 057103 (2005).
  - [4] W. Lenz, Z. Phys. **21**, 613 (1920);

- E. Ising, Z. Phys. **31**, 253 (1925);  
M. Hasenbusch, Int. J. Mod. Phys. C **12**, 911 (2001).
- [5] J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman, *A theory of critical phenomena. An Introduction to the renormalization group*. (Clarendon Press, Oxford, 1992).
  - [6] P. R. Campos, V. M. Oliveira, and F. G. B. Moreira, Phys. Rev. E **67**, 026104 (2003).
  - [7] L. F. C. Pereira and F. G. B. Moreira, Phys. Rev. E **71**, 016123 (2005).
  - [8] F. W. S. Lima, U. L. Fulco, and R. N. C. Filho, Phys. Rev. E **71**, 036105 (2005).
  - [9] F. W. S. Lima, Int. J. Mod. Phys. C **17** (2006), [e-print [arXiv:physics/0511082](#)].
  - [10] P. Erdős and A. Rényi, Publications Mathematicae **6**, 290 (1959);  
P. Erdős and A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
  - [11] R. Albert and A.-L. Barabási, Science **286**, 509 (1999).
  - [12] A. Aleksiejuk, J. A. Holyst, and D. Stauffer, Physica A **310**, 260 (2002).
  - [13] F. W. S. Lima, J. E. Moreira, J. S. Andrade, Jr., and U. M. S. Costa, Physica A **283**, 100 (2000);  
F. W. S. Lima, U. M. S. Costa, M. P. Almeida, and J. S. Andrade, Jr., Eur. Phys. J. B **17**, 111 (2000).
  - [14] D. Stauffer, A. Aharony, *Introduction to Percolation Theory*, (Taylor and Francis, London 1994);  
M. Sahimi, *Applications of Percolation Theory*, (Taylor and Francis, London 1994).
  - [15] P. N. Suding, R. M. Ziff, Phys. Rev. E **60**, 275 (1999).
  - [16] K. Malarz, M. Zborek, and B. Wróbel, TASK Quarterly **9**, 475 (2005).
  - [17] G. Grinstein, C. Jayaprakash, Yu He, Phys. Rev. Lett. **55**, 2527 (1985).
  - [18] K. Binder and D. W. Heermann, *Monte Carlo Simulation in Statistical Physics*, (Springer Verlag, 1988).
  - [19] R. Albert, A.-L. Barabási, Rev. Mod. Phys. **286**, 47 (2002);  
S. N. Dorogovtsev, J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002);  
M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
  - [20] The linear dimension of such networks, i.e. its diameter — defined as an average node-to-node distance — grows usually logarithmically with the system size [19].